Introduction to Mathematics and Modeling

lecture 2
First order differential equations

academic year : 18-19
lecture : 2
build : February 4, 2019
slides : 20

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This week

1. Section 7.2: separable differential equations
2. Section 9.4: autonomous differential equations
A **separable differential equation** is an equation of the form

\[
\frac{dy}{dx} = f(x)g(y)
\]

Solution method:

1. Separate the equation:
   \[
   \frac{1}{g(y)} \, dy = f(x) \, dx.
   \]

2. Then integrate:
   \[
   \int \frac{1}{g(y)} \, dy = \int f(x) \, dx.
   \]

3. (If possible) solve for \( y \).
Example 1

\[
\frac{dy}{dx} = (1 + y)e^x, \quad y > -1
\]

- Separate:
- Integrate:
- Solve \( y \):
Example 2

\[ \frac{dy}{dx} = -\frac{x}{y} \]
Separable differential equations: example 2

\[
\frac{dy}{dx} = -\frac{x}{y}
\]

- Separate:
- Integrate:
- Solve \( y \)?
A system is subject to **exponential change** if it is described by the differential equation

\[ y' = \alpha y \]

where \( \alpha \) is a constant.

- If \( \alpha > 0 \) then we the system is subject to **exponential growth**.
- If \( \alpha < 0 \) then we talk about **exponential decay**.
- The differential equation is separable.
Exponential change

\[ \frac{dy}{dx} = \alpha y. \]

- Separate:

- Integrate:

- Solve \( y \):

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Exponential change

\[
\frac{dy}{dx} = \alpha y \quad \Rightarrow \quad |y| = M e^{\alpha x}, \quad M > 0.
\]

- Since \(|y|\) is either \(+y\) or \(-y\), we can replace this equation by

\[
y = L e^{\alpha x},
\]

where \(L = \pm M\) is a non-zero constant.

- Separation fails to find the solution \(y(x) = 0\), but it certainly is a solution, so if we define \(K = L\) or \(K = 0\) then

\[
y(x) = K e^{\alpha x}, \quad K \in \mathbb{R}.
\]

- The constant \(K = y(0)\) is the initial value of \(y\):

\[
y(x) = y(0) e^{\alpha x}.
\]
Exponential change

\[ y' = \alpha y, \quad \alpha > 0 \]

Exponential growth

\[ y' = \alpha y, \quad \alpha < 0 \]

Exponential decay
An **autonomous differential equation** is a differential equation of the form

\[ y' = f(y) \]

- The slope field does not depend on \( x \).
- Along horizontal lines all line segments have the same slope.
- Solution curves can be shifted in horizontal direction.
- If \( y(x) \) is a solution, then \( y(x + C) \) is also a solution.
- If \( x \) represents time:

  Solutions of autonomous differential equations are **time independent**.
Example

\[ \frac{dy}{dx} = (y + 1)(y - 2) \]
Example

\[
\frac{dy}{dx} = (y + 1)(y - 2)
\]
Solution curves of autonomous differential equations can be sketched qualitatively with just a few computations.

1. **Find the equilibrium solutions:** solve \( f(y) = 0 \) for \( y \).

2. **Draw a phase line:**
   - On the \( y \)-axis mark the values for which \( f(y) = 0 \).
   - Identify the intervals where \( f(y) > 0 \) (with an arrow pointing upward: ↑) and \( f(y) < 0 \) (with an arrow pointing downward: ↓).

3. **Sketch some solutions in the \( xy \)-plane.**

The book also computes the sign of \( y'' \) to determine convexity of solutions. You don’t have to be able to do this!
Example 1

\[ \frac{dy}{dx} = (y + 1)(y - 2) \]
- Phase lines can also be drawn horizontally.
- If \( f(y) = 0 \), then \( y \) is an equilibrium point.
- If \( f(y) > 0 \), draw an arrow pointing to the right: \( \rightarrow \), and \( f(y) < 0 \) draw an arrow pointing leftward: \( \leftarrow \).
An equilibrium is called **asymptotically stable** if the arrows point towards the equilibrium point.

An equilibrium is called **unstable** if the arrows away from the equilibrium point.

If $f'(y_0) < 0$, then the equilibrium $y_0$ is stable, and if $f'(y_0) > 0$, then the equilibrium $y_0$ is unstable.
Example 2

\[ \frac{dy}{dx} = y^2 - 4 \]
Example 2

\[ \frac{dy}{dx} = y^2 - 4 \quad \Rightarrow \quad f(y) = y^2 - 4 \]

The equilibrium points are \(-2\) and \(2\).

- \( f'(y) = \)
- \( f'(-2) = \)
- \( f'(2) = \)
Example 2

\[
\frac{dy}{dx} = y^2 - 4 \quad \Rightarrow \quad f(y) = y^2 - 4
\]

- The equilibrium points are $-2$ and $2$.

- $f'(y) =$

- $f'(-2) =$

- $f'(2) =$
Example 2

\[
\frac{dy}{dx} = y^2 - 4 \implies f(y) = y^2 - 4, \quad f'(y) = 2y.
\]

- The equilibrium points are \(-2\) and \(2\).

- \(f'(y) = \)

- \(f'(-2) = \)

- \(f'(2) = \)
■ $u(t)$ is the voltage over the capacitor.

■ From Ohm's law one derives the following differential equation:

$$RC\ u' + u = V(t)$$

■ If $V(t) = V_B$ is constant then the equation is autonomous.
\[ RC \, u' + u = V_B \quad \Rightarrow \quad u' = \frac{V_B - u}{RC} \]

- \( f(u) = \frac{V_B - u}{RC} \), the equilibrium solution is \( u = V_B \).

- \( f'(V_B) = -\frac{1}{RC} < 0 \), the equilibrium \( V_B \) is stable.